Signal Analysis

Lecture #7

5CT.1-2,4

Homework

- Mean Squared Error
  - Problem (1)
    - For our example in class, prove that $E^2=0.017$ for $f(t) = \cos(\pi t/2)$
  - Problem (2)
    - It is desired to approximate $f(t) = \sin(t)$ in the interval $0<t<\pi/2$ by the straight line $f_s(t)=mt+b$. Determine the values of m and b for a least mean square error approximation and calculate the corresponding MSE.
- Fourier Series
  - Problem (3)
    - Compute the Fourier Series for the function using 3 terms in the series:

      , $f(t) = 1$ for $0<t<\pi, f(t)=0$ for $\pi<t<2\pi$
  - Problem (4)
    - Compute the Fourier Series for the function using 4 terms in the series:

      $f(t) = t$ for $0<t<3$
- 5CT.1.1, 5CT.1.2

Homework Answers #1

- Mean Squared Error
  - Problem (1)
    - For our example in class, prove that $E^2=0.017$ for $f(t) = cos\ (\pi t/2)$

$$\varepsilon(t) = f(t) - (a_0 + a_1 t + a_2 t^2) \text{ over the interval } -1 < \text{t} < +1$$

$$E^2 = \frac{1}{2}\int_{-1}^{+1}[f(t)]^2 dt - \int_{-1}^{+1}(a_0 + a_1 t + a_2 t^2) f(t) dt + \frac{1}{2}\int_{-1}^{+1}(a_0 + a_1 t + a_2 t^2)^2 dt$$

Chose a_i's to satisfy the minimum criteria for E^2

$$\frac{\partial E^2}{\partial a_k} = 0, \frac{\partial^2 E^2}{\partial a_k{}^2} > 0$$

Homework Answers #2

$$\frac{\partial E^2}{\partial a_o} = -\int_{-1}^{+1} f(t)dt + \frac{1}{2}\frac{\partial}{\partial a_o}[\int_{-1}^{+1}(a_0 + a_1 t + a_2 t^2)^2 dt = -\int_{-1}^{+1} f(t)dt + 2a_0 + \frac{2}{3}a_2 = 0$$

$$\frac{\partial E^2}{\partial a_1} = -\int_{-1}^{+1} tf(t)dt + \frac{1}{2}\frac{\partial}{\partial a_1}[\int_{-1}^{+1}(a_0 + a_1 t + a_2 t^2)^2 dt = -\int_{-1}^{+1} tf(t)dt + \frac{2}{3}a_1 = 0$$

$$\frac{\partial E^2}{\partial a_2} = -\int_{-1}^{+1} t^2 f(t)dt + \frac{1}{2}\frac{\partial}{\partial a_2}[\int_{-1}^{+1}(a_0 + a_1 t + a_2 t^2)^2 dt = -\int_{-1}^{+1} t^2 f(t)dt + \frac{2}{3}a_0 + \frac{2}{5}a_2 = 0$$

$$\int_{-1}^{+1} f(t)dt = \int_{-1}^{+1} \cos\frac{\pi t}{2}dt = \frac{2}{\pi}\sin\frac{\pi t}{2}\Big|_{-1}^{+1} = \frac{2}{\pi}(1 + (-1)) = \frac{4}{\pi}$$

$$\int_{-1}^{+1} tf(t)dt = \int_{-1}^{+1} t\cos\frac{\pi t}{2}dt$$

$$u = t, du = dt$$

$$dv = \cos\frac{\pi t}{2}dt, v = \frac{2}{\pi}\sin\frac{\pi t}{2}$$

$$\int_{-1}^{+1} t\cos\frac{\pi t}{2}dt = t\frac{2}{\pi}\sin\frac{\pi t}{2}\Big|_{-1}^{+1} - \int_{-1}^{+1}\frac{2}{\pi}\sin\frac{\pi t}{2}dt = \frac{2}{\pi}[t\sin\frac{\pi t}{2}\Big|_{-1}^{+1} - \{-\frac{2}{\pi}\cos\frac{\pi t}{2}\Big|_{-1}^{+1}\}]$$

$$= \frac{2}{\pi}[\{(1)(1) - (-1)(-1)\} - \{-\frac{2}{\pi}(0 - 0)\}] = 0$$

Homework Answers #3

$$\int_{-1}^{+1} t^2 \cos\frac{\pi t}{2} dt$$

$$u = t^2, du = 2t dt$$

$$dv = \cos\frac{\pi t}{2} dt, v = \frac{2}{\pi}\sin\frac{\pi t}{2}$$

$$\int_{-1}^{+1} t^2 \cos\frac{\pi t}{2} dt = t^2 \frac{2}{\pi}\sin\frac{\pi t}{2}\Big|_{-1}^{+1} - \frac{4}{\pi}\int_{-1}^{+1} t\sin\frac{\pi t}{2} dt = \frac{2}{\pi}(\{[1^2(1)] - [(-1)^2(-1)]\} - \frac{4}{\pi}\int_{-1}^{+1} t\sin\frac{\pi t}{2} dt = \frac{4}{\pi} - \frac{4}{\pi}\int_{-1}^{+1} t\sin\frac{\pi t}{2} dt$$

$$\int_{-1}^{+1} t\sin\frac{\pi t}{2} dt$$

$$u = t, du = dt$$

$$dv = \sin\frac{\pi t}{2} dt, v = -\frac{2}{\pi}\cos\frac{\pi t}{2}$$

$$\int_{-1}^{+1} t\sin\frac{\pi t}{2} dt = -\frac{2}{\pi}[t\cos\frac{\pi t}{2}\Big|_{-1}^{1} - \int_{-1}^{+1}\cos\frac{\pi t}{2} dt] = -\frac{2}{\pi}[t\cos\frac{\pi t}{2}\Big|_{-1}^{1} - (\frac{2}{\pi}\sin\frac{\pi t}{2}\Big|_{-1}^{1})]$$

$$= -\frac{2}{\pi}[\{1(0) - (-1)(0)\} - (\frac{2}{\pi}\{1 - (-1)\})] = \frac{8}{\pi^2}$$

$$\frac{4}{\pi} - \frac{4}{\pi}\int_{-1}^{+1} t\sin\frac{\pi t}{2} dt = \frac{4}{\pi} - \frac{4}{\pi}(\frac{8}{\pi^2}) = \frac{4}{\pi} - \frac{32}{\pi^3}$$

Homework Answers #4

$$\frac{\partial E^2}{\partial a_o} \Rightarrow 2a_0 + \frac{2}{3}a_2 = \frac{4}{\pi}$$

$$\frac{\partial E^2}{\partial a_1} \Rightarrow \frac{2}{3}a_1 = 0$$

$$\frac{\partial E^2}{\partial a_2} \Rightarrow \frac{2}{3}a_0 + \frac{2}{5}a_2 = \frac{4}{\pi} - \frac{32}{\pi^3}$$

$$a_0 = \frac{60}{\pi^3} - \frac{3}{\pi} = .98$$

$$a_1 = 0$$

$$a_2 = \frac{15}{\pi} - \frac{180}{\pi^3} = -1.03$$

$$f(t) = .98 - 1.03t^2$$

$\varepsilon(t) = f(t) - (.98 - 1.03t^2)$ over the interval $-1 < \text{t} < +1$

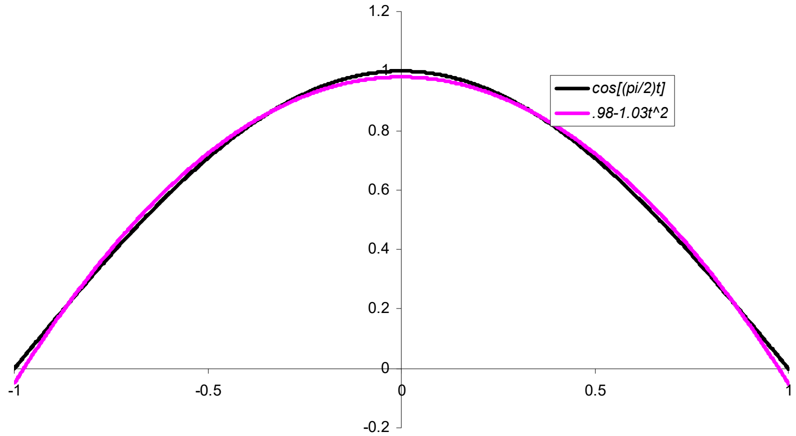
$$E^2 = \frac{1}{2}\int_{-1}^{+1}[\cos\frac{\pi}{2}t]^2 dt - \int_{-1}^{+1}(.98 - 1.03t^2)\cos\frac{\pi}{2}t dt + \frac{1}{2}\int_{-1}^{+1}(.98 - 1.03t^2)^2 dt$$

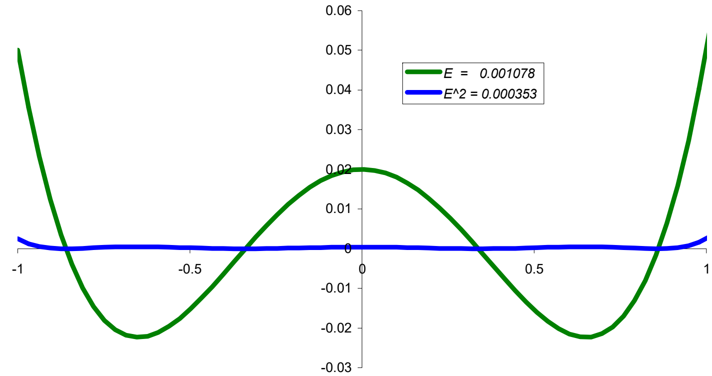
$$= \frac{1}{2} - \int_{-1}^{+1}(.98\cos\frac{\pi}{2}t - 1.03t^2\cos\frac{\pi}{2}t)dt + \frac{1}{2}\int_{-1}^{+1}[.98^2 - 2(.98)(1.03)t^2 + (1.03)^2 t^4]dt$$

$$= \frac{1}{2} - [.98(\frac{4}{\pi}) - 1.03(\frac{4}{\pi} - \frac{32}{\pi^3})] + \frac{1}{2}[.98^2(2) - \frac{2(.98)(1.03)(+1^3 - (-1)^3)}{3} + \frac{(1.03)^2(+1^5 - (-1)^5)}{5}]$$

$$= 0.5 - 1.2478 + 0.2484 + 0.499647 = .0003$$

Homework Answers #5





$$\varepsilon(t) = \cos\frac{\pi}{2}t - (.98 - 1.03t^2) \text{ over the interval } -1 < \mathrm{t} < +1$$

$$E^2 = .0003$$

Homework Answers #6

- Mean Squared Error
  - Problem (2)
    - It is desired to approximate *f(t) = sin (t)* in the interval *0<t<π/2* by the straight line *fa(t)=mt+b*. Determine the values of *m* and *b* for a least mean square error approximation and calculate the corresponding MSE.

$$E^2 = \frac{1}{\pi/2}\int_0^{+\pi/2}(f(t)-[b+mt])^2\,dt = \frac{2}{\pi}\int_0^{+\pi/2}([f(t)]^2 - 2[b+mt]f(t) + [b+mt]^2)dt$$

$$\frac{\partial E^2}{\partial b} = -\frac{4}{\pi}\int_0^{+\pi/2} f(t)dt + \frac{2}{\pi}\frac{\partial}{\partial b}\int_0^{+\pi/2}(b+mt)^2\,dt = -\frac{4}{\pi}\int_0^{+\pi/2} f(t)dt + \frac{4}{\pi}\int_0^{+\pi/2}(b+mt)dt$$

$$= -\frac{4}{\pi}\int_0^{+\pi/2} f(t)dt + \frac{4}{\pi}bt + \frac{4}{\pi}\frac{m}{2}t^2\Big|_0^{\pi/2} = -\frac{4}{\pi}\int_0^{+\pi/2} f(t)dt + 2b + \frac{m}{2}\pi = 0$$

$$\frac{\partial^2 E^2}{\partial b^2} = \frac{4}{\pi}\frac{\partial}{\partial b}\int_0^{+\pi/2}(b+mt)dt = 2 > 0$$

$$\frac{\partial E^2}{\partial m} = -\frac{4}{\pi}\int_0^{+\pi/2} tf(t)dt + \frac{2}{\pi}\frac{\partial}{\partial m}[\int_0^{+\pi/2}(b+mt)^2\,dt = -\frac{4}{\pi}\int_0^{+\pi/2} tf(t)dt + \frac{4}{\pi}\int_0^{+\pi/2}(b+mt)tdt$$

$$= -\frac{4}{\pi}\int_0^{+\pi/2} tf(t)dt + \frac{4}{\pi}\int_0^{+\pi/2}(bt+mt^2)dt = -\frac{4}{\pi}\int_0^{+\pi/2} tf(t)dt + \frac{b}{2}\pi + \frac{m}{6}\pi^2 = 0$$

$$\frac{\partial^2 E^2}{\partial m^2} = \frac{4}{\pi}\frac{\partial}{\partial m}\int_0^{+\pi/2}(b+mt)tdt = \frac{4}{\pi}\int_0^{+\pi/2} t^2 dt = \frac{\pi^2}{6} > 0$$

Homework Answers #7

$$\frac{\partial E^2}{\partial b} = -\frac{4}{\pi}\int_0^{+\pi/2} \sin t dt + 2b + \frac{m}{2}\pi = 0$$

$$2b + \frac{m}{2}\pi = \frac{4}{\pi}$$

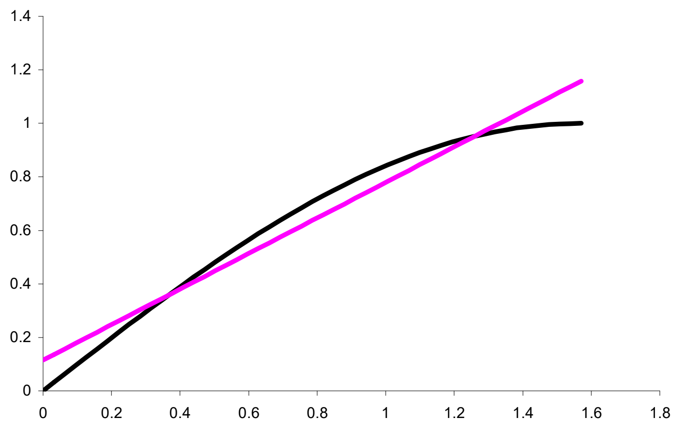
$$\frac{\partial E^2}{\partial m} = -\frac{4}{\pi}\int_0^{+\pi/2} t \sin t dt + \frac{b}{2}\pi + \frac{m}{6}\pi^2$$

$$\frac{b}{2}\pi + \frac{m}{6}\pi^2 = \frac{4}{\pi}$$

$$b = \frac{24}{\pi^2}(\frac{\pi}{3} - 1) = .115$$

$$m = \frac{48}{\pi^3}(2 - \frac{\pi}{2}) = .664$$

$$fa(t) = .115 + .664t$$



$$\int_0^{+\pi/2} \sin t dt = -\cos t\Big|_0^{+\pi/2} = -(0 - (1)) = 1$$

$$\int_0^{+\pi/2} t \sin t dt$$

$$u = t, du = dt$$

$$dv = \sin t dt, v = -\cos t$$

$$\int_0^{+\pi/2} t \sin t dt = [t(-\cos t)\Big|_0^{+\pi/2} - \int_0^{+\pi/2} (-\cos t) dt]$$

$$= [-t\cos t\Big|_0^{\pi/2} + (\sin t\Big|_0^{\pi/2})]$$

$$= [-\{\frac{\pi}{2}(0) - (0)(1)\} + (\{1 - (0)\})] = 1$$

$$E^2 = \frac{2}{\pi}\int_0^{+\pi/2} ([f(t)]^2 - 2[b + mt]f(t) + [b + mt]^2) dt$$

$$= \frac{2}{\pi}\int_0^{+\pi/2} ([\sin t]^2 - 2[.115 + .664t]\sin t + [.115 + .664t]^2) dt$$

$$= .187$$

Homework Answers #8

- Fourier Series
  - Problem (3)
    - Compute the Fourier Series for the function using 3 terms in the series:

      $f(t) = 1$ for $0<t<\pi, f(t)=0$ for $\pi<t<2\pi$

$$\mathbf{D}_k = \frac{1}{2\pi}\int_0^{2\pi} f(t)e^{-jkt}dt = \frac{1}{2\pi}\int_0^{\pi} 1e^{-jkt}dt = (\frac{1}{2\pi})(\frac{1}{-jk})\,e^{-jkt}\Big|_0^{\pi} = \frac{1}{-2\pi kj}(e^{-jk\pi}-1)$$

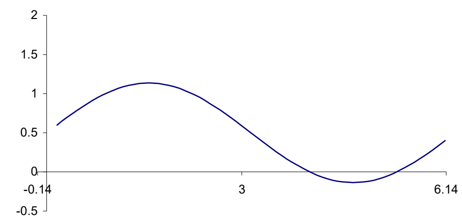
$$== \frac{1}{2\pi kj}(1-e^{-jk\pi}) = \frac{1}{2\pi kj}(1-(-1)^k)$$

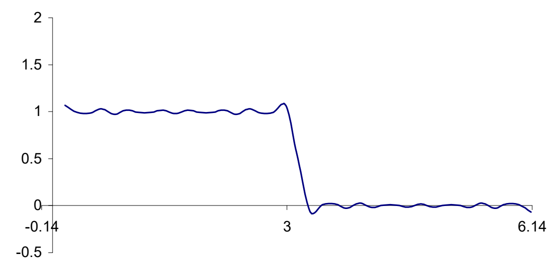
$$= \frac{1}{\pi k}e^{-j\pi/2};\ \text{for } k \neq 0; k \text{ odd}$$

$$=0;\ \text{for } k \neq 0; k \text{ even}$$

$$\mathbf{D}_0 = \frac{1}{2\pi}\int_0^{2\pi} f(t)dt = \frac{1}{2\pi}\int_0^{\pi} 1dt = \frac{1}{2}$$

$$f(t) = \frac{1}{2} + 2\sum_1^2 \frac{1}{\pi k}\cos(kt - \frac{\pi}{2})$$





Homework Answers #9

- Fourier Series
  - Problem (4)
    - Compute the Fourier Series for the function using 4 terms in the series:

      f(t) = t for *0<t<3*

$$\mathbf{D}_k = \frac{1}{3}\int_0^3 f(t)e^{-j2\pi kt/3}dt = \frac{1}{3}\int_0^3 te^{-j2\pi kt/3}dt$$

$$u = t, du = dt$$

$$dv = e^{-j2\pi kt/3}dt, v = \frac{1}{-j2\pi k/3}e^{-j2\pi kt/3}$$

$$\frac{1}{3}\int_0^\pi te^{-j2\pi kt/3}dt = \frac{1}{3}[\frac{t}{-j2\pi k/3}e^{-j2\pi kt/3}|_0^3 - \int_0^3 (\frac{1}{-j2\pi k/3})e^{-j2\pi kt/3}dt]$$

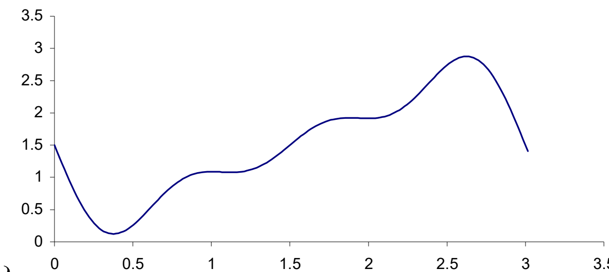
$$= \frac{1}{3}[\frac{1}{-j2\pi k/3}\{3e^{-j2\pi k} - 0\} - (\frac{1}{-j2\pi k/3})(\frac{1}{-j2\pi k/3}\{e^{-j2\pi k} - 1\}$$

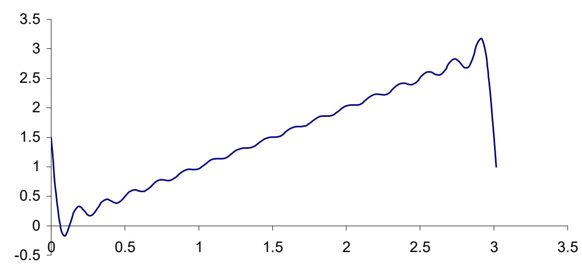
since $e^{-j2\pi k} = \cos(2\pi k) - j\sin(2\pi k) = 1 - j0$

$$\mathbf{D}_k = \frac{1}{-j2\pi k/3} = j\frac{3}{2\pi k}; k \neq 0$$

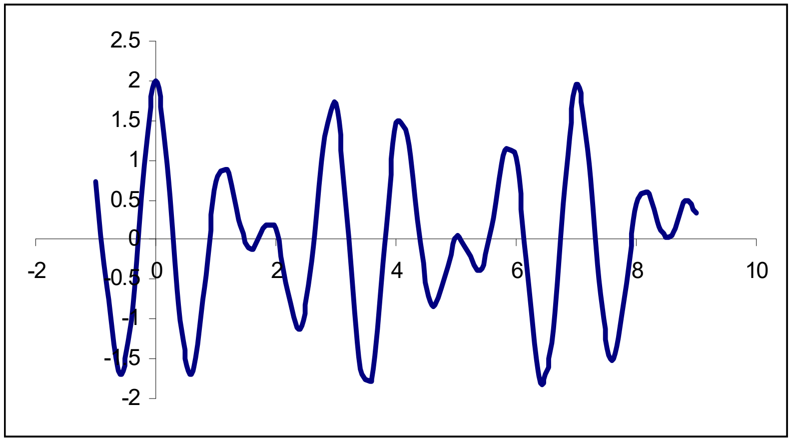
$$\mathbf{D}_0 = \frac{1}{3}\int_0^3 f(t)dt = \frac{1}{3}\int_0^3 tdt = (\frac{1}{3})\frac{9}{2} = 1.5$$

$$f(t) = 1.5 + 2\sum_1^3 \frac{3}{2\pi k}\cos(2\pi kt/3 + \pi/2)$$





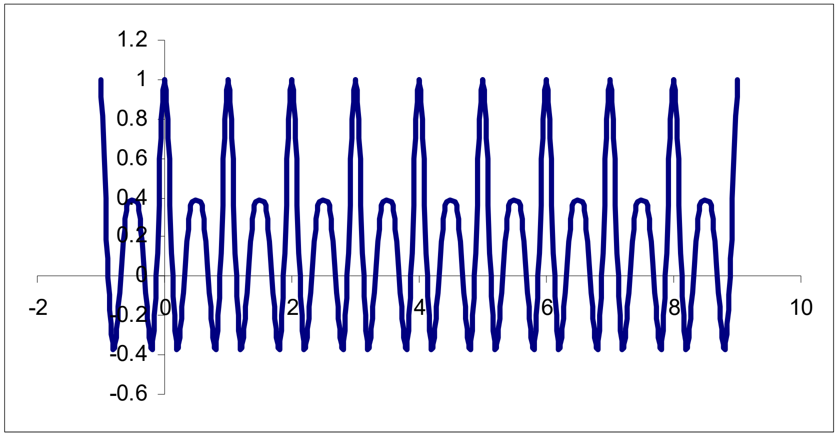
5CT.1.1a



$$x(t) = \cos(2\pi t) + \cos(\sqrt{2}\pi t)$$

Not Periodic: Frequencies are not integer multiples of each other

5CT.1.1b



$x(t) = \cos(2\pi\cos(2\pi t))$

Periodic: $\cos(2\pi t)$ is periodic with period 1 sec $\cos(2\pi\cos(2\pi t))$ transforms periodic function to new values

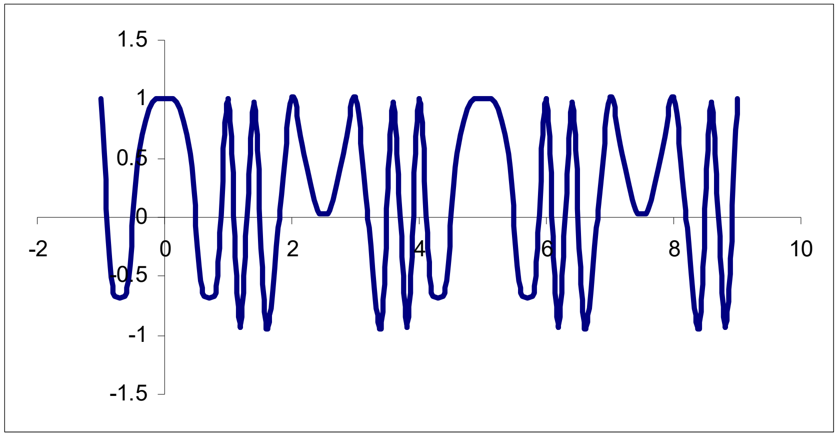
Proof:

$$\cos(\cos(2\pi\cos(2\pi t)) \overset{?}{=} \cos(\cos(2\pi\cos(2\pi[t-1])) = \cos(\cos(2\pi\cos(2\pi t - 2\pi 1]))$$

$$\cos(\cos(2\pi\cos(2\pi t)) \overset{?}{=} \cos(\cos(2\pi\cos(2\pi t - 2\pi))$$

$$\cos(\cos(2\pi\cos(2\pi t)) \overset{YES}{=} \cos(\cos(2\pi\cos(2\pi t))$$

5CT.1.1c



$x(t) = \cos(2\pi t^2)$

$$\cos(2\pi t^2) \stackrel{?}{=} \cos(2\pi[t-T]^2) = \cos(2\pi(t^2 - 2tT + T^2)) = \cos(2\pi t^2 - 4\pi tT + 2\pi T^2)$$

RECALL:

$\cos(A+B) = \cos A \cos B - \sin A \sin B$

$$\cos(2\pi t^2) \stackrel{?}{=} \cos(2\pi t^2 - 4\pi tT + 2\pi T^2) = \cos 2\pi t^2 \cos(2\pi[-2tT + T^2]) - \sin 2\pi t^2 \sin(2\pi[-2tT + T^2])$$

To be true $\cos(2\pi[-2tT + T^2]) = 1$ and $\sin 2\pi t^2 \sin(2\pi[-2tT + T^2]) = 0$

OR

$-2tT + T^2$ must be an integer which is impossible to for all values of t.

5CT.1.2

$$x(t) = 15 + 6.2\cos(300\pi t) + 8\sin(700\pi t)$$

$$x(t) = 15 + 6.2\cos(300\pi t) + 8\cos(700\pi t - \frac{\pi}{2})$$

$$x(t) = 15 + 6.2\cos([150]2\pi t) + 8\cos([350]2\pi t - \frac{\pi}{2})$$

$$x(t) = 15 + 6.2\cos(3[50]2\pi t) + 8\cos(7[50]2\pi t - \frac{\pi}{2})$$

Fundamental=50Hz

3rd Harmonic=150 Hz